UFIFT-HEP-94-4

# Stretching Wiggly Strings

Jaewan Kim and Pierre Sikivie

*Department of Physics, University of Florida, Gainesville, FL 32611*

(February 1, 2008)



## Abstract

How does the amplitude of a wiggle on a string change when the string is stretched? We answer this question for both longitudinal and transverse wiggles and for arbitrary equation of state, *i.e.*, for arbitrary relation between the tension $\tau$ and the energy per unit length $\epsilon$ of the string. This completes our derivation of the renormalization of string parameters which results from averaging out small scale wiggles on a string. The program is presented here in its entirety.

Typeset using REVTEX

## I. INTRODUCTION

In this paper, we give a general treatment of the renormalization of string parameters (the energy per unit length, the tension, and the equation of state) when a coarse grained view of a wiggly string is adopted. Although the formalism developed below may have applications in other areas, our primary motivation has been to try and improve our understanding of cosmic gauge strings [1]. Cosmic gauge strings are a prediction of certain grand unified theories. They may be present in the universe today as thin line-like regions of space which are hung up in a false vacuum of very high energy density. If one were to discover such strings, one would have the opportunity of learning physics in an energy regime which is completely out of reach of present earth-based laboratory experiments. As such, cosmic gauge strings are worthy of careful theoretical analysis.

A gauge string is an example of a Nambu-Goto string, *i.e.* it has energy per unit length $\epsilon$ equal to its tension $\tau$: $\epsilon = \tau \equiv \mu$, at the microscopic level. On the other hand, computer simulations have shown that cosmic gauge strings acquire small scale structure [2,3]. The effect of small scale structure is to increase the energy per unit length of a gauge string and to decrease its tension when a coarse grained description of the wiggly string is adopted [4,5]. Thus whereas the "bare" Nambu-Goto string has $\epsilon = \tau \equiv \mu$, the renormalized wiggly Nambu-Goto (RWNG) string, *i.e.* the coarse grained description of a wiggly Nambu-Goto string, has $\bar{\epsilon} > \mu$ and $\bar{\tau} < \mu$. As a consequence, the RWNG string does not obey the Nambu-Goto equations of motion.

In ref. [6] were derived the relativistic equations of motion for general strings, *i.e.* strings with arbitrary relation between the tension $\tau$ and the energy per unit length $\epsilon$, including RWNG strings. In addition, the renormalization of $\tau$ and $\epsilon$ that results from averaging out small scale wiggles on a string was obtained in the general case to lowest order in the amount of wiggliness. Let us state this latter result more precisely as follows.

Consider a string with energy per unit length $\epsilon$ and tension $\tau$. Suppose that the string has wiggles on it with characteristic wavelength $\lambda$ but that the string is otherwise straight.

Then an observer with resolution $d \gg \lambda$ will see a straight string, assigning to it an energy per unit length $\bar{\epsilon} > \epsilon$ and tension $\bar{\tau} \neq \tau$. $\bar{\epsilon}$ and $\bar{\tau}$ were calculated to lowest order in the amount of wiggliness. The result has the general form:

$$\bar{\epsilon} = \epsilon + C_\epsilon(\epsilon, \tau, \frac{d\tau}{d\epsilon}, \frac{d^2\tau}{d\epsilon^2}, \dots)\langle v^2 \rangle \tag{1.1a}$$

$$\bar{\tau} = \tau + C_\tau(\epsilon, \tau, \frac{d\tau}{d\epsilon}, \frac{d^2\tau}{d\epsilon^2}, \dots)\langle v^2 \rangle \tag{1.1b}$$

where $\langle v^2 \rangle$ is the average (velocity)$^2$ which characterizes the wiggles. The coefficients $C_\epsilon$ and $C_\tau$ can be found in ref. [6]. As indicated in Eqs. (1.1), these coefficients depend solely upon the equation of state $\tau(\epsilon)$ of the original string. The equation of state of a string gives the relationship between $\epsilon$ and $\tau$ when the string is slowly stretched. It must be derived from the relevant microphysics.

The main purpose of the present paper is to derive the equation of state $\bar{\tau}(\bar{\epsilon})$ of *the new string*, which is the coarse grained description of the wiggly string we started from. To do so we must obtain the response of $\langle v^2 \rangle$ to a slow stretching of the original wiggly string. Once the equation of state $\bar{\tau}(\bar{\epsilon})$ of the new string has been obtained we can compute $\frac{d\bar{\tau}}{d\bar{\epsilon}}$, $\frac{d^2\bar{\tau}}{d\bar{\epsilon}^2}$, ..., and insert these into Eqs. (1.1) if we wish to average out wiggles on the new string. The process of averaging out wiggles may then be repeated indefinitely, moving to successively longer scales and producing "running" $\epsilon(k)$, $\tau(k)$, $\frac{d\tau}{d\epsilon}(k)$, *etc.* These are the quantities describing the string when all wiggles of wavelength shorter than $\lambda = \frac{2\pi}{k}$ have been averaged over.

The plan of the paper is as follows. In section II, we obtain the relativistic equations of motion for strings with arbitrary equation of state. They are a generalization of the Nambu-Goto equations of motion. These equations were already obtained in ref. [6] but we repeat the derivation here for the sake of completeness and clarity. In section III, we obtain the renormalization of $\epsilon$ and $\tau$ which results from averaging out wiggles on the string. For the case of transverse wiggles the treatment and the results are the same as in ref. [6]. For the case of longitudinal wiggles, we adopt a new and definition of what it means to add a wiggle to a string. This changes the results for the renormalization of $\epsilon$ and $\tau$ from averaging

out longitudinal wiggles compared to those in ref. [6]. We will discuss our motivation for adopting the new definition. In section IV, we obtain the solution of the equations of motion which describes a homogeneously stretching string. In section V, we study the behaviour of small perturbations away from that solution. This will enable us to answer the question that motivated us in the first place: how does the amplitude of a wiggle change when the string is stretched? In section VI, we use the results to complete our derivation of the renormalization of string parameters from averaging out small scale wiggles. In section VII, we apply our results to the important special case of wiggly Nambu-Goto strings.

## II. EQUATIONS OF MOTION FOR GENERAL STRINGS

Consider a string, *i.e.*, an object whose stress-energy-momentum is localized on a line in space. Let $X^\mu(\sigma)$ be the space-time coordinates of the worldsheet with respect to a Lorentz reference frame. $\sigma = (\sigma^0, \sigma^1)$ are arbitrarily chosen coordinates parametrizing points on the worldsheet. The metric induced on the worldsheet is

$$h_{ab}(\sigma) = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(x), \qquad a, b = 0, 1\,, \tag{2.1}$$

where $g_{\mu\nu}$ is the space-time metric. For the purpose of this paper we will restrict ourselves to the case $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. At point $X^\mu(\sigma)$ on the string lives the 2-dim. stress-energy-momentum tensor

$$t^{ab} = (\epsilon - \tau) u^a u^b + \tau\, h^{ab}, \tag{2.2}$$

where $\epsilon(\sigma)$ is the energy per unit length of the string, $\tau(\sigma)$ is its tension and $u^a(\sigma)$ is the fluid velocity parallel to the string: $u^a u_a = +1$. The 4-dim. stress-energy-momentum tensor is then

$$T^{\mu\nu}(x) = \int d^2\sigma \sqrt{-h}\, t^{ab}(\sigma) \partial_a X^\mu(\sigma) \partial_b X^\nu(\sigma) \delta^4\Big(x - X(\sigma)\Big)\,, \tag{2.3}$$

where $h = \det(h_{ab})$. This expression is both invariant under 2-dim. reparametrizations and covariant under 4-dim. Lorentz transformations. The motion of the string must conserve 4-dim. energy and momentum. Thus

$$
\begin{aligned}
0 = \partial_\mu T^{\mu\nu}(x) &= \int d^2\sigma \sqrt{-h}\, t^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\mu \delta^4\Big(x - X(\sigma)\Big) \\
&= \int d^2\sigma \sqrt{-h}\, t^{ab} \partial_b X^\nu \partial_a \delta^4\Big(x - X(\sigma)\Big) \\
&= -\int d^2\sigma\, \partial_a \Big[\sqrt{-h} t^{ab} \partial_b X^\nu\Big] \delta^4\Big(x - X(\sigma)\Big).
\end{aligned} \tag{2.4}
$$

The equations of motion for general strings are therefore

$$
\partial_a \Big[\sqrt{-h}\, t^{ab}(\sigma) \partial_b X^\mu\Big] = 0 \quad (\mu = 0, 1, 2, 3)\ . \tag{2.5}
$$

To complete the description of the string dynamics, we must supplement Eq. (2.5) with an equation of state:

$$
\tau = \tau(\epsilon)\ . \tag{2.6}
$$

We then have five equations for the five unknowns: $\vec{X}_\perp(\sigma)$, $\epsilon(\sigma)$, $\tau(\sigma)$, and $\beta(\sigma) \equiv \frac{u^1(\sigma)}{u^0(\sigma)}$. $\vec{X}_\perp(\sigma)$ represents the two transverse degrees of freedom of the string. $\beta(\sigma)$ is its longitudinal velocity. The five equations uniquely specify the time evolution that results from arbitrary initial conditions. It was shown in ref. [6] that the case of the Nambu-Goto string is included in this description. It corresponds to the equation of state $\epsilon = \tau$.

Eqs. (2.5) and (2.6) describe how the fluid motion along the string affects the motion of the string in the transverse directions, and vice versa. This dynamics is purely classical. However, there are also interesting quantum mechanical effects associated with the addition of new degrees of freedom to a string. Indeed, consider a straight string at rest. From a classical viewpoint, there are in this case no forces of the string upon the fluid attached to it. However, from a quantum mechanical viewpoint, there are forces of the string upon the fluid because the fluid distribution affects the quantum mechanical fluctuations of the string. In ref. [7], these forces were calculated for the case of beads attached to a straight string. In the present paper, such order $\hbar$ effects are neglected.

## III. RENORMALIZATION OF $\epsilon$ AND $\tau$ DUE TO WIGGLINESS

In this section, we derive in detail the renormalization of the energy per unit length and the tension of a wiggly string which must be performed when a coarse grained description

of the string is adopted. Consider first a straight motionless string lying along the $x$-axis. This is a trivial static solution to Eqs. (2.5). It is given in the $\sigma = (t, x)$ gauge by

$$X^\mu(\sigma) = (t, x, 0, 0) \tag{3.1a}$$

$$\epsilon(t, x) = \epsilon_0 = \text{constant} \tag{3.1b}$$

$$\tau(t, x) = \tau(\epsilon_0) = \tau_0 \tag{3.1c}$$

$$\beta(t, x) = 0. \tag{3.1d}$$

Next we perturb the string, exciting small oscillations about the static solution. In a coarse grained view, the string appears straight and motionless again, but with the 4-dim. stress-energy-momentum tensor:

$$\overline{T}^{\mu\nu} = \text{diag}(\bar{\epsilon}, -\bar{\tau}, 0, 0)\delta(y)\delta(z) = \langle T^{\mu\nu} \rangle, \tag{3.2}$$

where $T^{\mu\nu}$ is the stress-energy-momentum tensor of the wiggly string and $\langle T^{\mu\nu} \rangle$ is the space-time average thereof. We have assumed, as we do throughout this paper, that the wiggles do not carry average momentum with respect to the Lorentz frame in which the unperturbed string is at rest. In order to compute $\langle T^{\mu\nu} \rangle$, it is necessary to solve the linearized equations of motion for the wiggles with appropriate initial conditions.

What are the appropriate initial conditions? Equivalently, one might ask: what does it mean to add a wiggle to a string? Throughout this paper we will adopt the following definition. Consider a length $L$ of string whose unperturbed state is given by Eqs. (3.1). At time $t = 0$, apply sudden infinite (impulse type) forces to the string which instantaneously change its local longitudinal and transverse velocities without changing any positions. After $t = 0$, let the string move freely. It can be shown that, for the purpose of the present paper, that definition is equivalent to the following one. Consider a length $L$ of string whose unperturbed state is given by Eq. (3.1). Stretch this string longitudinally and/or transversely keeping the position of the end points fixed in the laboratory frame of reference. Then let the string move freely.

For transverse wiggles in the $y$-direction, the initial condition consistent with our definition is

$$X^{\mu}(0,x) = (0,x,0,0) \tag{3.3a}$$

$$\partial_t X^{\mu}(0,x) = (1,0,\dot{y}_{\mathrm{in}}(x),0) \tag{3.3b}$$

$$\beta(0,x) = 0 \tag{3.3c}$$

$$\epsilon(0,x) = \epsilon_0 \tag{3.3d}$$

$$\tau(0,x) = \tau_0, \tag{3.3e}$$

where $\dot{y}_{\mathrm{in}}(x)$ is the initial transverse velocity at point $x$. Eqs. (3.3) are the initial conditions for transverse wiggles used in ref. [6]. For longitudinal wiggles, the initial condition is

$$X^{\mu}(0,x) = (0,x,0,0) \tag{3.4a}$$

$$\partial_t X^{\mu}(0,x) = (1,0,0,0) \tag{3.4b}$$

$$\beta(0,x) = \beta_{\mathrm{in}}(x) \tag{3.4c}$$

$$\epsilon(0,x) = \epsilon_0 - \frac{1}{2}(\epsilon_0 - \tau_0){\beta_{\mathrm{in}}}^2(x) + \mathcal{O}({\beta_{\mathrm{in}}}^4) \tag{3.4d}$$

$$\tau(0,x) = \tau(\epsilon(0,x)), \tag{3.4e}$$

where $\beta_{\mathrm{in}}(x)$ is the initial longitudinal velocity at point $x$. Eqs. (3.4) are not the initial conditions for longitudinal wiggles used in ref. [6]. In ref. [6], we set $\epsilon(0,x) = \epsilon_0$ and $\tau(0,x) = \tau_0$. We will give the reasons for the change in section V. With the definition adopted in this paper of what it means to add a wiggle to a string, we must use Eqs. (3.4). Indeed, consider a small length $\Delta L$ of string at point $x$. At time $t = 0$, since positions do not change instantaneously, that length does not change in the laboratory frame of reference. Therefore, because of Lorentz contraction, in the local instantaneous rest frame of the string after $t = 0$, that small piece of string has length

$$\Delta L[1 - {\beta_{\mathrm{in}}}^2(x)]^{-1/2} = \Delta L[1 + \frac{1}{2}{\beta_{\mathrm{in}}}^2(x)] + \mathcal{O}({\beta_{\mathrm{in}}}^4). \tag{3.5}$$

In its rest frame, the piece of string got stretched. The stretching of the proper length $\Delta L$

by the amount $d\Delta L = \frac{1}{2}{\beta_{\rm in}}^2(x)\Delta L$ produces a decrease $d\epsilon$ in the energy per unit length which is readily obtained by equating the change in energy with the work done:

$$d(\epsilon\Delta L) = \tau d\Delta L \tag{3.6}$$

which yields

$$d\epsilon = -(\epsilon-\tau)\frac{d\Delta L}{\Delta L} = -\frac{1}{2}(\epsilon-\tau)\beta_{\rm in}^2, \tag{3.7}$$

which in turn yields Eq. (3.4d). Eq. (3.4e) then follows from the equation of state.

We are now ready to proceed. Let us first analyze longitudinal wiggles. The longitudinally perturbed string is described in the $\sigma = (t,x)$ gauge by

$$X^\mu(t,x) = (t,x,0,0) \tag{3.8a}$$

$$\beta(t,x) = \beta_1(t,x) + \beta_2(t,x) + \ldots \tag{3.8b}$$

$$\epsilon(t,x) = \epsilon_0 + \epsilon_1(t,x) + \epsilon_2(t,x) + \ldots \tag{3.8c}$$

$$\tau(t,x) = \tau_0 + \tau_1(t,x) + \tau_2(t,x) + \ldots\ , \tag{3.8d}$$

where the expansions are in powers of $\beta_{\rm in}$. The equation of state relates $\tau$ and $\epsilon$. Hence

$$\tau_1 = +\frac{d\tau}{d\epsilon}\Big|_0\epsilon_1 = -v_{L,0}^2\epsilon_1, \tag{3.9a}$$

$$\tau_2 = -v_{L,0}^2\epsilon_2 + \frac{1}{2}\frac{d^2\tau}{d\epsilon^2}\Big|_0\epsilon_1^2, \tag{3.9b}$$

where $v_{L,0} \equiv \sqrt{-\frac{d\tau}{d\epsilon}|_0}$. The 2-dim. stress-energy-momentum tensor is

$$\left(t^{ab}\right) = \begin{pmatrix} (\epsilon-\tau)\gamma^2+\tau & (\epsilon-\tau)\gamma^2\beta \\ (\epsilon-\tau)\gamma^2\beta & (\epsilon-\tau)\gamma^2\beta^2-\tau \end{pmatrix}, \tag{3.10}$$

where $\gamma = (1-\beta^2)^{-1/2}$. The linearized equations of motion are

$$\dot{\epsilon_1} + (\epsilon_0-\tau_0)\beta_1' = 0 \tag{3.11a}$$

$$(\epsilon_0-\tau_0)\dot{\beta}_1 + v_{L,0}^2\epsilon_1' = 0, \tag{3.11b}$$

where dots and primes denote derivatives with respective to $t$ and $x$ respectively. If the longitudinal oscillation is initiated by giving longitudinal velocity $\beta_{\rm in}(x) = \beta_{\rm in}\sin kx$ at time $t=0$, then the solution to Eqs. (3.11) and (3.4) is

$$\beta_1(t,x) = \beta_{\rm in} \sin kx \, \cos v_{L,0} kt \tag{3.12a}$$

$$\epsilon_1(t,x) = -(\epsilon_0 - \tau_0) \frac{\beta_{\rm in}}{v_{L,0}} \cos kx \, \sin v_{L,0} kt. \tag{3.12b}$$

Eqs. (3.12) show that, as was implied by our notation, $v_{L,0}$ is the phase velocity of longitudinal wiggles. $\langle T^{\mu\nu} \rangle$ must be computed up to second order in $\beta_{\rm in}$. Using Eqs. (2.3), (3.2), (3.4), (3.9), and the fact that $\langle \epsilon_1 \rangle = \langle \beta_1 \rangle = 0$, we find

$$\bar{\epsilon} = \langle t^{00} \rangle = \epsilon_0 + \langle \epsilon_2 \rangle + (\epsilon_0 - \tau_0) \langle \beta_1^2 \rangle \tag{3.13a}$$

$$0 = \langle t^{01} \rangle = (\epsilon_0 - \tau_0) \langle \beta_2 \rangle + (1 + v_{L,0}^2) \langle \epsilon_1 \beta_1 \rangle \tag{3.13b}$$

$$-\bar{\tau} = \langle t^{11} \rangle = -\tau_0 + (\epsilon_0 - \tau_0) \langle \beta_1^2 \rangle + v_{L,0}^2 \langle \epsilon_2 \rangle - \frac{1}{2} \frac{d^2 \tau}{d\epsilon^2}\Big|_0 \langle \epsilon_1^2 \rangle. \tag{3.13c}$$

Eq. (3.13b) expresses our assumption that there is no net flow of momentum in either direction. To determine $\langle \epsilon_2 \rangle$ we use the equation of motion in the time-like direction, to second order in $\beta_{\rm in}$:

$$\dot{\epsilon}_2 + 2(\epsilon_0 - \tau_0) \beta_1 \dot{\beta}_1 + (\epsilon_0 - \tau_0) \beta_2' + (1 + v_{L,0}^2) [\epsilon_1 \beta_1]' = 0. \tag{3.14}$$

Averaging Eq. (3.14) over an interval $0 \leq x \leq L$, we obtain

$$\partial_t \frac{1}{L} \int_0^L dx [\epsilon_2 + (\epsilon_0 - \tau_0) \beta_1^2] = \mathcal{O}(\frac{1}{L}). \tag{3.15}$$

This equation merely tells us that the average energy associated with the longitudinal perturbation is time-independent. So we may evaluate the average energy from the initial condition alone, using Eqs. (3.4d) and (3.12a):

$$\begin{aligned} \langle \epsilon_2 \rangle + (\epsilon_0 - \tau_0) \langle \beta_1^2 \rangle &= \frac{1}{L} \int_0^L dx \Big[ \epsilon_2 + (\epsilon_0 - \tau_0) \beta_1^2 \Big]_{t=0} \\ &= \frac{1}{4} (\epsilon_0 - \tau_0) \beta_{\rm in}^2 = (\epsilon_0 - \tau_0) \langle \beta_1^2 \rangle. \end{aligned} \tag{3.16}$$

Hence

$$\langle \epsilon_2 \rangle = 0. \tag{3.17}$$

The renormalized energy per unit length and tension due to longitudinal perturbations are therefore

$$\bar{\epsilon}_L = \epsilon_0 + (\epsilon_0 - \tau_0)\langle\beta_1^2\rangle \tag{3.18a}$$

$$\bar{\tau}_L = \tau_0 - (\epsilon_0 - \tau_0)\langle\beta_1^2\rangle\Big[1 + (\epsilon_0 - \tau_0)\frac{d\ln v_L}{d\epsilon}\Big|_0\Big], \tag{3.18b}$$

to lowest non-trivial order in the amount of wiggliness. The amount of longitudinal wiggliness is characterized in our formalism by $\langle\beta_1^2\rangle$. Eqs. (3.18) differ from the corresponding equations in ref. [6] because we use here a different definition of what is meant by adding a wiggle to the string. The inadequacy of the definition used in ref. [6] did not become apparent to us until we studied the behaviour of the wiggles when the string is being stretched (see section V).

Next let us consider transverse wiggles. In the gauge $\sigma = (t, x)$, the space-time coordinates of a wiggly string whose average direction is along the $x$-axis is

$$(X^\mu(t,x)) = (t, x, y(t,x), z(t,x)), \tag{3.19}$$

where $y(t,x)$ and $z(t,x)$ average to zero. The induced metric of the worldsheet is

$$(h_{ab}) = \begin{pmatrix} 1 - \dot{y}^2 - \dot{z}^2 & -\dot{y}y' - \dot{z}z' \\ -\dot{y}y' - \dot{z}z' & -1 - y'^2 - z'^2 \end{pmatrix}. \tag{3.20}$$

Let us write

$$\epsilon(t,x) = \epsilon_0 + \epsilon_1(t,x) \tag{3.21a}$$

$$\tau(t,x) = \tau_0 - v_{L,0}^2\epsilon_1(t,x) + \mathcal{O}(\epsilon_1^2). \tag{3.21b}$$

We will see below that $\epsilon_1$ and $\beta$ are of second order in $y$ and $z$. The 2-dim. stress-energy-momentum tensor up to second order in $y$ and $z$ is then

$$(t^{ab}) = \begin{pmatrix} \epsilon_0 + \epsilon_1 + \epsilon_0(\dot{y}^2 + \dot{z}^2) & (\epsilon_0 - \tau_0)\beta - \tau_0(\dot{y}y' + \dot{z}z') \\ (\epsilon_0 - \tau_0)\beta - \tau_0(\dot{y}y' + \dot{z}z') & -\tau_0 + v_{L,0}^2\epsilon_1 + \tau_0(y'^2 + z'^2) \end{pmatrix}. \tag{3.22}$$

Because of rotational symmetry around the $x$-axis and the lack of mixing between the $y$- and $z$-components of the wiggles up to second order, we set $z = 0$ and focus solely on the $y$-component from hereon. Up to second order in $y$, the equations of motion are

$$\epsilon_0 \ddot{y} - \tau_0 y'' = 0 \tag{3.23a}$$

$$\dot{\epsilon}_1 + (\epsilon_0 - \tau_0)\beta' + (\epsilon_0 - \tau_0)\dot{y}'y' = 0 \tag{3.23b}$$

$$(\epsilon_0 - \tau_0)\dot{\beta} + v_{L,0}^2 \epsilon_1' + \tau_0 y'(y'' - \ddot{y}) = 0. \tag{3.23c}$$

Eq. (3.23a) tells us that transverse wiggles are sinusoidal waves with phase velocity $v_{T,0} = \sqrt{\frac{\tau_0}{\epsilon_0}}$. Eqs. (3.23b) and (3.23c) may be combined to produce wave equations for $\epsilon_1$ and $\beta$ with source terms:

$$\ddot{\epsilon}_1 - v_{L,0}^2 \epsilon_1'' = -\frac{1}{2}[\epsilon_0 \partial_t^2 - \tau_0 \partial_x^2](\dot{y}^2 + y'^2) \tag{3.24a}$$

$$\ddot{\beta} - v_{L,0}^2 \beta'' = \frac{v_{L,0}^2 - v_{T,0}^2}{2(1 - v_{T,0}^2)} \partial_t \partial_x (\dot{y}^2 + y'^2) + \frac{v_{T,0}^2}{1 - v_{T,0}^2}[\partial_t^2 - v_{L,0}^2 \partial_x^2]\dot{y}y'. \tag{3.24b}$$

These equations show that transverse wiggles induce perturbations in $\epsilon$ and $\beta$ which are second order in $y$.

Let us initiate the transverse wiggle by imposing the initial conditions (3.3) with $\dot{y}_{\rm in}(x) = \dot{y}(0, x) = y_{\rm in} v_{T,0} k \sin kx$. Then Eq. (3.23a) implies

$$y(t, x) = y_{\rm in} v_{T,0} k \sin kx \sin v_{T,0} kt. \tag{3.25}$$

From Eq. (3.23b), we obtain

$$\partial_t \frac{1}{L}\int_0^L dx[\epsilon_1 + \frac{1}{2}(\epsilon_0 - \tau_0)y'^2] = \mathcal{O}(\frac{1}{L}). \tag{3.26}$$

Hence

$$\begin{aligned} &\langle \epsilon_1 \rangle + \frac{1}{2}(\epsilon_0 - \tau_0)\langle y'^2 \rangle \\ &= \frac{1}{L}\int_0^L dx \Big[\epsilon_1 + \frac{1}{2}(\epsilon_0 - \tau_0)y'^2)\Big]_{t=0} = 0, \end{aligned} \tag{3.27}$$

since $y'(0, x) = \epsilon_1(0, x) = 0$. Therefore,

$$\begin{aligned} \langle \epsilon_1 \rangle &= -\frac{1}{2}(\epsilon_0 - \tau_0)\langle y'^2 \rangle \\ &= -\frac{1}{2}(\epsilon_0 - \tau_0)\frac{\epsilon_0}{\tau_0}\langle \dot{y}^2 \rangle \end{aligned} \tag{3.28}$$

From Eqs. (2.3), (3.2), (3.22) and (3.28) we obtain

$$\bar{\epsilon}_T = \langle\sqrt{-h}t^{00}\rangle = \epsilon_0 + \epsilon_0(\langle\dot{y}^2\rangle + \langle\dot{z}^2\rangle) \tag{3.29a}$$

$$\bar{\tau}_T = -\langle\sqrt{-h}t^{11}\rangle = \tau_0 - \frac{1}{2}(\langle\dot{y}^2\rangle + \langle\dot{z}^2\rangle)[\tau_0 + \epsilon_0 + v_{L,0}^2\epsilon_0(1-\frac{\epsilon_0}{\tau_0})], \tag{3.29b}$$

for the renormalized energy per unit length and tension from averaging out small scale transverse wiggles. Combining Eqs. (3.18) and (3.29), the renormalization group equations (RGE) for $\epsilon(k)$ and $\tau(k)$ are

$$-\frac{d\epsilon}{d\ln k} = W_T(k)\epsilon + W_L(k)(\epsilon-\tau) \tag{3.30a}$$

$$\begin{aligned}-\frac{d\tau}{d\ln k} &= -\frac{1}{2}W_T(k)\Big[\tau + \epsilon + v_L^2\epsilon(1-\frac{\epsilon}{\tau})\Big]\\ &\quad -W_L(k)(\epsilon-\tau)\Big[1+(\epsilon-\tau)\frac{d\ln v_L}{d\epsilon}\Big],\end{aligned} \tag{3.30b}$$

where $W_T(k)$ and $W_L(k)$ are the spectral densities on a $\ln k$ scale of $\langle\dot{y}^2\rangle + \langle\dot{z}^2\rangle$ and $\langle\beta^2\rangle$ respectively.

However, as was already emphasized in ref. [6] and also in the introduction, Eqs. (3.30) are not by themselves complete since the equation of state, as given by $v_L^2 = -\frac{d\tau}{d\epsilon}$, $\frac{d\ln v_L}{d\epsilon} = -\frac{1}{2v_L^2}\frac{d^2\tau}{d\epsilon^2}$, *etc.*, is also modified by the wiggles and hence varies from scale to scale. To obtain the modification of the equation of state from averaging out small scale wiggles we must study the response of the wiggles to an adiabatic stretching of the string.

## IV. HOMOGENEOUS STRETCHING SOLUTIONS

In this section we find the solution of the equations of motion which describes a straight string which is stretching uniformly. Later we will add wiggles onto this background solution and we will study how the wiggles' amplitudes respond to the adiabatic stretching.

Our homogeneous stretching ansatz is that the gradient of $\epsilon$, and hence the gradient of $\tau$ as well, be everywhere parallel to the local 2-velocity:

$$\partial_a\epsilon = C(\sigma)u_a. \tag{4.1}$$

Using $u^a u_a = 1$, one readily obtains

$$(h^{ab} - u^a u^b)\, \partial_b \epsilon = 0, \tag{4.2}$$

as an equivalent expression.

Let us assume that the string is lying along the $x$-axis. In the $\sigma = (t, x)$ gauge, Eq. (4.1) is equivalent to

$$\partial_x \epsilon + \beta \partial_t \epsilon = 0. \tag{4.3}$$

Also in this gauge the equations of motion (2.5) become simply

$$\partial_a t^{ab} = 0, \quad b = 0, 1\,, \tag{4.4}$$

where $t^{ab}$ is given by Eq. (2.2) with $(h^{ab}) = (\eta^{ab}) = \mathrm{diag}(1, -1)$.

The component of Eqs. (4.4) which is perpendicular to $u^a$ [*i.e.*, $(h^c_b - u^c u_b)\partial_a\, t^{ab} = 0$] and Eq. (4.2) together imply

$$u^c \partial_c u^b = 0, \tag{4.5}$$

which is equivalent to the statement that the trajectory of any comoving point on the string has zero acceleration:

$$\frac{d\beta}{dt} = \partial_t \beta + \beta \partial_x \beta = 0. \tag{4.6}$$

This result can be easily understood. Indeed, because the string is homogeneously stretching, in the instantaneous rest frame of any physical point of the string, the string on both sides of the point appears the same (*i.e.*, the right hand side is the reflection of the left hand side) and hence there can be no force on the point by symmetry.

Consider the physical point which at time $t = 0$ is located at $x = x_0$. At later times, it is located at

$$x(t, x_0) = x_0 + t\beta_0(x_0), \tag{4.7}$$

where $\beta_0(x_0) = \beta(0, x_0)$. We have

$$\beta(t, x(t, x_0)) = \beta_0(x_0). \tag{4.8}$$

Taking the derivative of Eq. (4.8) with respect to $x_0$, we obtain

$$\partial_x \beta(t, x(t, x_0)) = \frac{H(x_0)}{1 + t\, H(x_0)}, \tag{4.9}$$

where $H(x_0) \equiv \frac{d\beta_0}{dx_0}$.

Next, let us analyze the component of the equations of motion which is parallel to $u^a$: $u^b \partial_a\, t^{ab} = 0$. This may be rewritten as

$$u^a \partial_a \epsilon = -(\epsilon - \tau) \partial_a u^a, \tag{4.10}$$

and then again as

$$\frac{d\epsilon}{dt} = \partial_t \epsilon + \beta \partial_x \epsilon = -(\epsilon - \tau) \partial_x \beta. \tag{4.11}$$

We may integrate Eq. (4.11) along the trajectory of any physical point on the string. Using Eq. (4.9), this yields

$$\int_{\epsilon(t_1, x(t_1, x_0))}^{\epsilon(0, x_0)} \frac{d\epsilon}{\epsilon - \tau(\epsilon)} = \int_0^{t_1} \frac{H(x_0)}{1 + t\, H(x_0)} dt = \ln \Big[ 1 + t_1\, H(x_0) \Big]. \tag{4.12}$$

Eq. (4.12) determines $\epsilon(t, x)$ in terms of the initial conditions $\epsilon(0, x_0)$ and $\beta(0, x_0) \equiv \beta_0(x_0)$. We now demand that the resulting $\epsilon(t, x)$ satisfy the homogeneous stretching ansatz (4.3). Taking the derivative of Eq. (4.12) with respect to $x_0$ and using Eq. (4.3), we obtain

$$\frac{d\epsilon(0, x_0)}{dx_0} = (\epsilon(x_0) - \tau(x_0)) \Big[ \frac{\beta_0(x_0) H(x_0)}{1 - \beta_0(x_0)^2} + \frac{t_1 \frac{dH}{dx_0}}{1 + t_1\, H(x_0)} \Big]. \tag{4.13}$$

This requires $\frac{dH}{dx_0} = 0$ or

$$\beta_0(x_0) = H x_0, \quad \text{where} \quad H = \text{constant}, \tag{4.14}$$

and

$$\frac{d\epsilon(0, x_0)}{dx_0} = \Big[ \epsilon(x_0) - \tau(x_0) \Big] \frac{H x_0}{1 - H^2 x_0^2}. \tag{4.15}$$

Eqs. (4.14) and (4.15) give the initial conditions which produce a homogeneous stretching solution. Note that the solution depends on only two parameters: $H$ and $\epsilon(0, 0)$. Note also

that the solution exists only in a limited region of space-time. At the initial time ($t = 0$), the solution exists only for $\mid x_0 \mid < H^{-1}$. At other times, the solution exist only for $\mid x \mid < H^{-1}+t$. See Fig. I.

This homogeneous stretching solution may be obtained in another, possibly more elegant, way. Let us choose coordinates $(\lambda, \sigma)$ such that a comoving point on the string is always labeled by the same space-like coordinate $\sigma$, and such that the direction of the time-like coordinate $\lambda$ is everywhere orthogonal to that of $\sigma$, in the Minkowskian sense. In this coordinate system, $u^1 = 0$ and hence the homogeneity condition (4.1) implies $\epsilon = \epsilon(\lambda)$. The induced metric is still diagonal however. Since $\epsilon$ does not depend upon $\sigma$, we may assume, as part of our homogeneity ansatz, that the metric also does not depend upon $\sigma$. We may then choose $\lambda$ in such a way that

$$(h_{ab}) = a^2(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.16}$$

which is equivalent to the following three conditions relating $(t, x)$ and $(\lambda, \sigma)$:

$$\dot{t}^2 - \dot{x}^2 = a^2(\lambda) \tag{4.17a}$$

$$\dot{t}t' - \dot{x}x' = 0 \tag{4.17b}$$

$$\dot{t'}^2 - \dot{x'}^2 = -a^2(\lambda) \tag{4.17c}$$

where the dots and primes denote derivatives with respect to $\lambda$ and $\sigma$ respectively. Eqs. (4.17) may be rewritten as

$$\frac{\partial(t, x)}{\partial(\lambda, \sigma)} = a(\lambda) \begin{pmatrix} \cosh\psi(\lambda, \sigma) & \sinh\psi(\lambda, \sigma) \\ \sinh\psi(\lambda, \sigma) & \cosh\psi(\lambda, \sigma) \end{pmatrix}, \tag{4.18}$$

where $\psi(\lambda, \sigma)$ and $a(\lambda)$ are as yet unknown functions. They are constrained by the commutativity of partial derivatives which implies

$$a\psi' \sinh\psi = \dot{a}\sinh\psi + a\dot{\psi}\cosh\psi \tag{4.19a}$$

$$a\psi' \cosh\psi = \dot{a}\cosh\psi + a\dot{\psi}\sinh\psi. \tag{4.19b}$$

These equations in turn imply

$$(a\psi' - \dot{a})^2 = a^2\dot{\psi}^2 \tag{4.20a}$$

$$a\dot{\psi}\tanh\psi = a\psi' - \dot{a}, \tag{4.20b}$$

which in turn imply

$$\psi' - \frac{\dot{a}}{a} = \dot{\psi}\tanh\psi = \pm\dot{\psi}. \tag{4.21}$$

The solution $\psi$ = constant, $a$ = constant, with $\tanh\psi = \pm 1$, does not yield an invertible metric and hence is not acceptable. The unique remaining solution is

$$\psi = H\sigma, \quad a = e^{H\lambda} \tag{4.22}$$

where $H$ is a constant. Substituting this back into Eqs. (4.18) and solving the corresponding partial differential equations for $t(\lambda, \sigma)$ and $x(\lambda, \sigma)$, we obtain

$$t = \frac{1}{H}(e^{H\lambda}\cosh H\sigma - 1) \tag{4.23a}$$

$$x = \frac{1}{H}e^{H\lambda}\sinh H\sigma, \tag{4.23b}$$

where the constants of integration were fixed by requiring the point $(t, x) = (0, 0)$ to be mapped onto the point $(\lambda, \sigma) = (0, 0)$. See Fig. I.

The 2-velocity in the $(\lambda, \sigma)$ coordinates is $(\frac{1}{a}, 0)$. By transforming back to the $(t, x)$ coordinates, we find

$$\beta(t, x) = \frac{u^1}{u^0}\Big|_{(t,x)} = \tanh H\sigma = \frac{Hx}{1 + Ht}, \tag{4.24}$$

which is consistent with our previous description [*cf.* Eqs. (4.9) and (4.14)]. The only independent equation of motion in $(\lambda, \sigma)$ coordinates is

$$\dot{\epsilon} = -(\epsilon - \tau)H. \tag{4.25}$$

Again, this is consistent with our previous description [*cf.* Eqs. (4.11) and (4.15)]. Eq. (4.25) determines $\epsilon(\lambda)$ for a given equation of state. For example, the renormalized wiggly Nambu-Goto string equation of state, $\epsilon\tau = \mu^2$ (see section VII), yields

$$\epsilon(\lambda) = \sqrt{\mu^2 + e^{-2H\lambda}(\epsilon_i^2 - \mu^2)}, \tag{4.26}$$

where $\epsilon_i$ is the energy per unit length on the $\lambda = 0$ hypersurface: $(1 + Ht)^2 - H^2x^2 = 1$.

## V. STRETCHING A WIGGLY STRING

We now add wiggles as small deviations away from the solution describing a homogeneously stretching string. In the comoving orthogonal gauge, $(\sigma^0, \sigma^1) \equiv (\lambda, \sigma)$, introduced in the previous section, the unperturbed string is described by

$$(X^\mu) = (t(\lambda, \sigma), x(\lambda, \sigma), 0, 0) \tag{5.1a}$$

$$\epsilon = \epsilon_0(\lambda) \tag{5.1b}$$

$$\tau = \tau(\epsilon_0(\lambda)) \equiv \tau_0(\lambda) \tag{5.1c}$$

$$(u^a) = (e^{-H\lambda}, 0). \tag{5.1d}$$

where $\epsilon_0(\lambda)$ is the solution of Eq. (4.25), and $t(\lambda, \sigma)$ and $x(\lambda, \sigma)$ are given in Eqs. (4.23).

Let us first discuss longitudinal wiggles. In that case, the perturbed string is described by

$$(X^\mu) = (t(\lambda, \sigma), x(\lambda, \sigma), 0, 0) \tag{5.2a}$$

$$\epsilon = \epsilon_0(\lambda) + \epsilon_1(\lambda, \sigma) \tag{5.2b}$$

$$\tau = \tau(\epsilon) = \tau_0(\lambda) - v_{L,0}^2(\lambda)\epsilon_1(\lambda, \sigma) + \mathcal{O}(\epsilon_1^2) \tag{5.2c}$$

$$(u^a) = \frac{e^{-H\lambda}}{\sqrt{1-\beta^2}}(1, \beta), \tag{5.2d}$$

with $t(\lambda, \sigma)$ and $x(\lambda, \sigma)$ still given by Eqs. (4.23), and $v_{L,0}^2(\lambda) = -\frac{d\tau}{d\epsilon}(\epsilon_0(\lambda))$. The stress-energy-momentum tensor is

$$(t^{ab}) = e^{-2H\lambda} \begin{pmatrix} (\epsilon - \tau)\gamma^2 + \tau & (\epsilon - \tau)\gamma^2\beta \\ (\epsilon - \tau)\gamma^2\beta & (\epsilon - \tau)\gamma^2\beta^2 - \tau \end{pmatrix}. \tag{5.3}$$

The equations of motion may be shown to be equivalent to

$$\partial_\lambda[(\epsilon - \tau)\gamma^2 + \tau] + \partial_\sigma[(\epsilon - \tau)\gamma^2\beta] + H(\epsilon - \tau)\gamma^2(1 + \beta^2) = 0 \tag{5.4a}$$

$$\partial_\lambda[(\epsilon - \tau)\gamma^2\beta] + \partial_\sigma[(\epsilon - \tau)\gamma^2\beta^2 - \tau] + 2H(\epsilon - \tau)\gamma^2\beta = 0. \tag{5.4b}$$

From Eqs. (5.4), one readily obtains the equations of motion first order in $\beta$ and $\delta \equiv \frac{\epsilon_1}{\epsilon_0}$:

$$\partial_\lambda \delta + H(v_{L,0}^2 + v_{T,0}^2)\delta + (1 - v_{T,0}^2)\partial_\sigma \beta = 0 \tag{5.5a}$$

$$(1 - v_{T,0}^2)\partial_\lambda \beta + H(1 - v_{T,0}^2)(1 - v_{L,0}^2)\beta + v_{L,0}^2 \partial_\sigma \delta = 0. \tag{5.5b}$$

Since the coefficients in Eqs. (5.5) do not depend upon $\sigma$, we may write

$$\beta(\lambda, \sigma) = \beta(\lambda) e^{ik\sigma}, \quad \delta(\lambda, \sigma) = \delta(\lambda) e^{ik\sigma}, \tag{5.6}$$

where $k$ is the wavevector in the comoving coordinates. We are interested in the regime where the wiggle is being stretched adiabatically. This requires $H \ll k$. Therefore we drop terms of order $H^2$, $H\frac{dv_{L,0}}{d\lambda}$, *etc.*, as we rearrange Eqs. (5.5) to separate $\delta$ and $\beta$:

$$\ddot{\delta} + H(1 + 2v_{T,0}^2 + v_{L,0}^2)\dot{\delta} + v_{L,0}^2 k^2 \delta = 0 \tag{5.7a}$$

$$\ddot{\beta} + [\frac{d}{d\lambda} \ln(\frac{1 - v_{T,0}^2}{v_{L,0}^2}) + H(1 + v_{T,0}^2)]\dot{\beta} + v_{L,0}^2 k^2 \beta = 0. \tag{5.7b}$$

Because the coefficients in Eqs. (5.7) are slowly varying functions of $\lambda$, we may solve these equations by the method of adiabatic invariants. This yields

$$\delta(\lambda, \sigma) = Ce^{-H\lambda}\Big[\frac{1 - v_{T,0}^2(\lambda)}{\epsilon_0(\lambda) v_{L,0}(\lambda)}\Big]^{1/2} e^{ik[\sigma \pm \int^\lambda d\lambda' v_{L,0}(\lambda')]} \tag{5.8a}$$

$$\beta(\lambda, \sigma) = -Ce^{-H\lambda}\Big[\frac{v_{L,0}(\lambda)}{(1 - v_{T,0}^2(\lambda))\epsilon_0(\lambda)}\Big]^{1/2} e^{ik[\sigma \pm \int^\lambda d\lambda' v_{L,0}(\lambda')]}, \tag{5.8b}$$

where $C$ is a constant. These equations tell us how the amplitude and the phase of a longitudinal wiggle change when the string is slowly stretched.

As was mentioned in section I and III, we modified in this paper the definition of what it means to add a wiggle to a string, from what it was in ref. [6]. We are finally in a position to explain what prompted the modification. With the definition used in ref. [6], one has, instead of Eq. (3.17),

$$\langle \epsilon_2 \rangle = (\epsilon_0 - \tau_0)\langle \beta^2 \rangle \tag{5.9}$$

for longitudinal wiggles. Hence there are in this case two equal contributions, $\langle \epsilon_2 \rangle$ and $(\epsilon_0 - \tau_0)\langle \beta^2 \rangle$, to the renormalization of $\epsilon_0$ (see Eq. (3.13)). However, when this string is stretched and the analysis presented above is applied to it, one finds that $\langle \epsilon_2 \rangle$ and $(\epsilon_0 - \tau_0)\langle \beta^2 \rangle$

do not remain equal during the stretching. Their rates of decrease are different. The reason for this phenomenon is that what was called "adding a longitudinal wiggle to a string" in ref. [6] actually has two different components. The first component is a homogeneous modification of the original string, and the second component is adding a wiggle. The two components behave differently under homogeneous stretching.

Next, let us consider transverse wiggles. The wiggly string is described by

$$(X^\mu) = (t(\lambda,\sigma), x(\lambda,\sigma), y(\lambda,\sigma), 0) \tag{5.10a}$$

$$\epsilon = \epsilon_0(\lambda) + \epsilon_1(\lambda,\sigma) \tag{5.10b}$$

$$\tau = \tau(\epsilon) = \tau_0(\lambda) - v_{L,0}^2 \epsilon_1(\lambda,\sigma) + \mathcal{O}(\epsilon_1^2) \tag{5.10c}$$

$$(h_{ab}) = e^{2H\lambda}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \dot{y}^2 & \dot{y}y' \\ \dot{y}y' & y'^2 \end{pmatrix} \tag{5.10d}$$

$$(u^a) = \frac{(1,\beta)}{\sqrt{e^{2H\lambda}(1-\beta^2) - (\dot{y}+\beta y')^2}}. \tag{5.10e}$$

The linearized equation of motion for y, which follows from the $\mu = 2$ component of Eqs. (2.5), is

$$\partial_\lambda[\epsilon_0(\lambda)\partial_\lambda\, y] - \tau_0(\lambda)\partial_\sigma^2\, y = 0\,. \tag{5.11}$$

Again we set $y(\lambda,\sigma) = y(\lambda)\, e^{ik\sigma}$ and use the method of adiabatic invariants to find

$$y(\lambda,\sigma) = \frac{C'}{[\epsilon(\lambda)\tau(\lambda)]^{1/4}} e^{ik[\sigma \pm \int^\lambda d\lambda' v_{T,0}(\lambda')]}. \tag{5.12}$$

Eq. (5.12) tells us how the amplitude and the phase of a transverse wiggle change when the string is slowly stretched.

## VI. RENORMALIZATION OF THE EQUATION OF STATE DUE TO WIGGLINESS

The equation of state function $\tau(\epsilon)$ is equivalent to the infinite set of parameters: $\epsilon$, $\tau$, $\frac{d\tau}{d\epsilon} \equiv -v_L^2$, $\frac{d^2\tau}{d\epsilon^2}$, $\frac{d^3\tau}{d\epsilon^3}$, $\ldots$. In ref. [6] and in section III of this paper, the renormalization of the

first two of these parameters, $\epsilon$ and $\tau$, which results from averaging out small scale wiggles on the string was obtained to lowest order in the amount of wiggliness. It has the form:

$$\bar{\epsilon} = \epsilon + C_\epsilon(\epsilon, \tau, v_L, \ldots)\langle v^2\rangle \tag{6.1a}$$

$$\bar{\tau} = \tau + C_\tau(\epsilon, \tau, v_L, \ldots)\langle v^2\rangle \tag{6.1b}$$

where $\langle v^2\rangle$ is the average (velocity)$^2$ associated with the wiggle. The coefficients $C_\epsilon$ and $C_\tau$ are given in Eqs. (3.18) and (3.29) for longitudinal and transverse wiggles respectively. In the previous section, we derived the response of the wiggles to a slow stretching of the string. We now use these results to derive, in principle, the renormalization of all the other parameters ($\frac{d^n\tau}{d\epsilon^n}$, $n = 1, 2, 3, \ldots$) that characterize the equation of state.

We wish to determine the coefficients $C_n$:

$$\frac{d^n\bar{\tau}}{d\bar{\epsilon}^n} = \frac{d^n\tau}{d\epsilon^n} + C_n\langle v^2\rangle + \mathcal{O}(v^4) \tag{6.2}$$

for $n = 1, 2, 3, \ldots$. What we will succeed in doing is to derive a recursion formula for these $C_n$. Let us call $\frac{1}{H}\frac{dX}{d\lambda}$ the rate of change of quantity $X$ under slow stretching. The notation $\frac{1}{H}\frac{d}{d\lambda}$ is taken from the previous section. We have

$$\frac{1}{H}\frac{d\epsilon}{d\lambda} = -(\epsilon - \tau) \tag{6.3a}$$

$$\frac{1}{H}\frac{d}{d\lambda}\frac{d^n\tau}{d\epsilon^n} = -(\epsilon - \tau)\frac{d^{n+1}\tau}{d\epsilon^{n+1}} \tag{6.3b}$$

$$\frac{1}{H}\frac{d}{d\lambda}C_n = -(\epsilon - \tau)\frac{dC_n}{d\epsilon}. \tag{6.3c}$$

Eq. (6.3a) is the same as Eq. (4.25). Eqs. (6.3b) and (6.3c) follow from the fact that, through the equation of state, $\tau$, $\frac{d^n\tau}{d\epsilon^n}$, $C_\epsilon$, $C_\tau$, $C_n$ are functions of $\epsilon$ only. The crucial information gained in the previous section is the rate of change of $\langle v^2\rangle$ under slow stretching. Let us define coefficients $D_v$ by

$$\frac{1}{H}\frac{d}{d\lambda}\langle v^2\rangle = -D_v\langle v^2\rangle. \tag{6.4}$$

Expressions for the coefficients $D_v$ for both transverse and longitudinal wiggles may be obtained from Eqs. (5.8b) and (5.12). They are given below. Suffice it to say for the

moment that the coefficients $D_v$, like the coefficients $C_n$, are functionals of the equation of state only. Combining Eqs. (6.1), (6.2), (6.3) and (6.4), we have

$$\frac{d^{n+1}\bar{\tau}}{d\bar{\epsilon}^{n+1}} = \frac{\frac{1}{H}\frac{d}{d\lambda}\frac{d^n\bar{\tau}}{d\bar{\epsilon}^n}}{\frac{1}{H}\frac{d}{d\lambda}\bar{\epsilon}} = \frac{\frac{d^{n+1}\tau}{d\epsilon^{n+1}} + \langle v^2\rangle[\frac{dC_n}{d\epsilon} + \frac{C_n D_v}{\epsilon-\tau}]}{1 + \langle v^2\rangle[\frac{dC_\epsilon}{d\epsilon} + \frac{C_\epsilon D_v}{\epsilon-\tau}]}. \tag{6.5}$$

Hence, the recursion formula:

$$C_{n+1} = \frac{dC_n}{d\epsilon} + \frac{C_n D_v}{\epsilon - \tau} - \frac{d^{n+1}\tau}{d\epsilon^{n+1}}\Big[\frac{dC_\epsilon}{d\epsilon} + \frac{C_\epsilon D_v}{\epsilon - \tau}\Big], \tag{6.6}$$

which allows one, in principle, to compute all the coefficients $C_n$ successively starting with $C_\tau \equiv C_0$. However, for a general equation of state, the coefficients $C_n$ quickly become very complicated.

For transverse wiggles, using Eqs. (5.12), (6.3a) and (6.3b) and the fact that $\frac{1}{H}\frac{d}{d\lambda}\ln k_{\rm phys} = -1$, we obtain

$$\begin{aligned}\frac{1}{H}\frac{d}{d\lambda}\langle\dot{y}^2\rangle &= \frac{1}{H}\frac{d}{d\lambda}[k^2_{\rm phys}v^2_T\langle y^2\rangle] \\ &= \langle\dot{y}^2\rangle[-2 + \frac{1}{2}(1 - \frac{\tau}{\epsilon})(3 + v^2_L)]\end{aligned} \tag{6.7}$$

where $k_{\rm phys}$ is the physical wavevector as opposed to the comoving wavevector $k$ introduced in Eq. (5.6). Thus for transverse wiggles, from Eqs. (3.29) and (6.7), we have

$$\langle v^2\rangle = \langle\dot{y}^2\rangle + \langle\dot{z}^2\rangle \tag{6.8a}$$

$$C_\epsilon = \epsilon \tag{6.8b}$$

$$C_\tau = C_0 = -\frac{1}{2}[\tau + \epsilon - \epsilon(1 - \frac{\epsilon}{\tau})\frac{d\tau}{d\epsilon}] \tag{6.8c}$$

$$D_v = +2 - \frac{1}{2}(1 - \frac{\tau}{\epsilon})(3 - \frac{d\tau}{d\epsilon}). \tag{6.8d}$$

These expressions may be fed into Eq. (6.6) to obtain the coefficients $C_n$ for transverse wiggles. The first one is

$$\begin{aligned}C_1 = \frac{1}{4} + \frac{3}{4}\frac{\tau}{\epsilon} - \frac{\tau+\epsilon}{\epsilon-\tau} - \frac{d\tau}{d\epsilon}\Big[\frac{1}{2} + \frac{1}{4}\frac{\epsilon}{\tau} + \frac{1}{4}\frac{\tau}{\epsilon} + \frac{(\epsilon+\tau)\epsilon}{\tau(\epsilon-\tau)}\Big] \\ + \frac{1}{2}\frac{d^2\tau}{d\epsilon^2}\epsilon(1 - \frac{\tau}{\epsilon}) + \frac{1}{2}\Big(\frac{d\tau}{d\epsilon}\Big)^2\Big[\frac{\epsilon^2}{\tau^2} - \frac{1}{2}(1 + \frac{\epsilon}{\tau})\Big].\end{aligned} \tag{6.9}$$

For longitudinal wiggles, using Eqs. (5.8b) and (6.3), we obtain

$$\begin{aligned}\frac{1}{H}\frac{d}{d\lambda}\langle\beta^2\rangle &= \frac{1}{H}\frac{d}{d\lambda}\Big[C^2 e^{-2H\lambda}\frac{v_L}{\epsilon(\lambda)-\tau(\lambda)}\Big] \\ &= \langle\beta^2\rangle\Big[-1+v_L^2-(\epsilon-\tau)\frac{d\ln v_L}{d\epsilon}\Big]. \end{aligned}\tag{6.10}$$

From Eqs. (3.13) and (6.10), we have then

$$\langle v^2\rangle = \langle\beta^2\rangle \tag{6.11a}$$

$$C_\epsilon = \epsilon-\tau \tag{6.11b}$$

$$C_\tau = -(\epsilon-\tau)\Big[1+(\epsilon-\tau)\frac{d\ln v_L}{d\epsilon}\Big] \tag{6.11c}$$

$$D_v = 1-v_L^2+(\epsilon-\tau)\frac{d\ln v_L}{d\epsilon}. \tag{6.11d}$$

These expressions may be fed into Eq. (6.6) to obtain the coefficients $C_n$ for longitudinal wiggles. The first one is

$$\begin{aligned}C_1 = -2-2\frac{d\tau}{d\epsilon}-4(\epsilon-\tau)\frac{d\ln v_L}{d\epsilon} \\ -(\epsilon-\tau)^2\Big(\frac{d\ln v_L}{d\epsilon}\Big)^2-(\epsilon-\tau)^2\frac{d^2\ln v_L}{d\epsilon^2}.\end{aligned}\tag{6.12}$$

## VII. WIGGLY NAMBU-GOTO STRINGS

In this section, we apply our formalism to the case of gauge strings or any other strings which at the microscopic level obey the Nambu-Goto (NG) equation of state $\epsilon=\tau=\mu$. In this case,

$$C_{\epsilon T}=\mu, \quad C_{\tau T}=-\mu \tag{7.1a}$$

$$D_{vT}=2 \tag{7.1b}$$

$$C_{\epsilon L}=C_{\tau L}=0, \tag{7.1c}$$

where the subscripts $T$ and $L$ stand for transverse and longitudinal respectively. Eq. (7.1c) reflects the fact that Nambu-Goto strings can not have longitudinal oscillations. They may

have transverse oscillations and Eq. (7.1a) tells us that the string parameters characterizing the renormalized wiggly Nambu-Goto (RWNG) string are

$$\bar{\epsilon} = \mu(1 + \langle \dot{y}^2 \rangle + \langle \dot{z}^2 \rangle) \tag{7.2a}$$

$$\bar{\tau} = \mu(1 - \langle \dot{y}^2 \rangle - \langle \dot{z}^2 \rangle) \tag{7.2b}$$

to lowest order in $\langle \dot{y}^2 \rangle$ and $\langle \dot{z}^2 \rangle$. From Eq. (7.1b), or more directly from Eq. (5.12) with $\epsilon_0 = \tau_0 = \mu$, we learn that if a wiggly Nambu-Goto string is stretched, the amplitude of the wiggle does not change. Vilenkin [1] had already established this fact in the case of a wiggly Nambu-Goto string that is being stretched by the expansion of the universe. A simple way to derive this result is as follows. Under homogeneous stretching, using Eqs. (7.2), we have:

$$\begin{aligned} \frac{1}{H}\frac{d\bar{\epsilon}}{d\lambda} &= -(\bar{\epsilon} - \bar{\tau}) = -2\mu(\langle \dot{y}^2 \rangle + \langle \dot{z}^2 \rangle) \\ &= \frac{\mu}{H}\frac{d}{d\lambda}(\langle \dot{y}^2 \rangle + \langle \dot{z}^2 \rangle). \end{aligned} \tag{7.3}$$

This implies

$$\begin{aligned} &\frac{1}{H}\frac{d}{d\lambda}\ln(\langle \dot{y}^2 \rangle + \langle \dot{z}^2 \rangle) \\ = &\frac{1}{H}\frac{d}{d\lambda}\ln[k_{\mathrm{phys}}^2(\langle y^2 \rangle + \langle z^2 \rangle)] = -2, \end{aligned} \tag{7.4}$$

and hence

$$\frac{1}{H}\frac{d}{d\lambda}(\langle y^2 \rangle + \langle z^2 \rangle) = 0 \tag{7.5}$$

since $\frac{1}{H}\frac{d}{d\lambda}\ln k_{\mathrm{phys}} = -1$. While the string is being stretched, $\bar{\epsilon}$ decreases and $\bar{\tau}$ increases in such a way as to keep $\bar{\epsilon}\bar{\tau} = \mu^2$. This is the equation of state of the RWNG string to the order to which we have calculated.

Now we add wiggles to the RWNG string. We may do that if the wavelength of the new wiggles are long compared to the wavelength of the wiggles that have already been averaged over. The equation of state that must be fed into Eqs. (3.18) and (3.29) to obtain the renormalized values of $\epsilon$ and $\tau$ which result from averaging out wiggles on the RWNG string is $\epsilon\tau = \mu^2$. But it now turns out, remarkably, that the equation of state $\epsilon\tau = \mu^2$ is

a fixed point of the renormalization group equations (3.30). Indeed one may readily verify that

$$\frac{d}{d\ln k}(\epsilon\tau) = 0 \tag{7.6}$$

for all $W_T(k)$ and $W_L(k)$ if the equation of state is $\epsilon\tau =$ constant. This means that $\epsilon\tau = \mu^2$ is the equation of state of renormalized wiggly Nambu-Goto strings no matter how many wiggles are added to the string providing only that $W_T(k)$ and $W_L(k)$, as functions of wavevector $k$, are everywhere sufficiently small. The equation of state $\epsilon\tau = \mu^2$ for renormalized wiggly Nambu-Goto strings was conjectured by Carter [4] and Vilenkin [5]. In ref. [6], using the argument just given, it was shown that $\epsilon\tau = \mu^2$ is indeed the equation of state of renormalized wiggly Nambu-Goto strings to lowest non-trivial order in the amount of wiggliness. But it was also explicitly shown in ref. [6] that in higher orders there are deviations from $\epsilon\tau = \mu^2$.

Fig. II provides a summary. The short-dashed lines represent the paths in $(\epsilon, \tau)$ space which are followed when a string is homogeneously stretched. The short-dashed lines are therefore given by the equation of state. The long-dashed lines represent the paths in $(\epsilon, \tau)$ space which are followed when one renormalizes $\epsilon$ and $\tau$ to average out small scale wiggles. The long-dashed lines are therefore given by integrating the renormalization group equations (3.30). The central aim of this paper is to derive the modification of the equation of state that results from averaging out small scale wiggles, *i.e.*, to determine how the short-dashed lines are related to one another when one moves along the long-dashed lines. Our results in this regard are stated in sections V and VI. The equation of state $\epsilon\tau = \mu^2$ is shown by the solid line in Fig. II. It has the following special property: if at any point on the solid line the short-dashed line is parallel to the solid line, then the long-dashed line is also parallel to the solid line there. As a consequence of this and the fact that $\epsilon\tau = \mu^2$ for a wiggly Nambu-Goto string near $\epsilon = \tau = \mu$, when adding wiggles to such a string and averaging over them, one remains forever on the $\epsilon\tau = \mu^2$ line, to lowest order in perturbation theory.

Finally, let us apply the results of section V to the RWNG string. Using the equation of state $\epsilon\tau = \mu^2$, Eq. (5.12) implies that the amplitude of transverse wiggles on a RWNG string

stays constant when the string is slowly stretched, just as is the case for the Nambu-Goto string. In addition, Eq. (5.8b) tells us that the $\beta$-amplitude of a longitudinal wiggle on a RWNG string is constant when the string is slowly stretched.

## ACKNOWLEDGMENTS

We are grateful to Jooyoo Hong for useful discussions. This work was supported in part by the U. S. Department of Energy under contract No. DE-FG05-86ER40272.

## REFERENCES

[1] For a review on cosmic strings, see A. Vilenkin, Phys. Rep. **121**, 263 (1985).

[2] D. P. Bennett and F. R. Bouchet, Phys. Rev. Lett. **60**, 257 (1988) and Phys. Rev. Lett. **63**, 2776 (1989) and in *Symposium on the formation and evolution of cosmic strings*, edited by G. W. Gibbons, S. W. Hawking, and T. Vachaspati (Cambridge Univ. Press, Cambridge, 1989); B. Allen and E.P.S. Shellard, Phys. Rev. Lett. **64**, 119 (1990).

[3] The generation and evolution of small scale structure on cosmic gauge strings are discussed in T. W. B. Kibble and E. Copeland, Phys. Scripta **T36**, 153 (1991); B. Allen and R. Caldwell, Phys. Rev. D **43**, 2457 (1991); E. Copeland, T. W. B. Kibble and D. Austin, Phys. Rev. D **48**, 5594 (1993); Phys. Rev. D **45**, R1000 (1992).

[4] B. Carter, Phys. Rev. D **41**, 3869 (1990).

[5] A. Vilenkin, Phys. Rev. D **41**, 3038 (1990).

[6] J. Hong, J. Kim, P. Sikivie, Phys Rev. Lett. **69**, 2611 (1992).

[7] E. D'Hoker and P. Sikivie, Phys. Rev. Lett. **71**, 1136 (1993).

## FIGURES

FIG. 1. Comoving Orthogonal Coordinates parametrizing a homogeneously stretching string: the radial lines which emerge from $(t, x) = (-\frac{1}{H}, 0)$ are equi-$\sigma$, while the hyperbolas are equi-$\lambda$. Note that the homogeneously stretching string may be defined only inside the future light cone of $(t, x) = (-\frac{1}{H}, 0)$.

FIG. 2. Behaviour of string parameters $\epsilon$ and $\tau$ under adiabatic stretching and under renormalization to average out small scale wiggles. The point of coordinates $(\epsilon, \tau)$ moves along a short-dashed line when the string is adiabatically stretching, whereas it moves along a long-dashed line when $\epsilon$ and $\tau$ are redefined to average out small scale wiggles. The solid line represents RWNG strings for which $\epsilon\tau = \mu^2$, while the blob at the end corresponds to a NG string.